particle in a 1D box

none

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Abstract

Generated by the Physics Derivation Graph.

Eq. 1 is an initial equation.

$$\frac{d^2}{dx^2}\psi(x) = -k^2\psi(x) \tag{1}$$

Judicious choice as a guessed solution to Eq. 1 is Eq. 2,

$$\psi(x) = a\sin(kx) + b\cos(kx) \tag{2}$$

A boundary condition for Eq. 1 is Eq. 3

$$\psi(x = 0) = 0 \tag{3}$$

A boundary condition for Eq. 1 is Eq. 4

$$\psi(x = W) = 0 \tag{4}$$

LHS of Eq. 3 is equal to LHS of Eq. 2; yields Eq. 5.

$$0 = a\sin(0) + b\cos(0) \tag{5}$$

Simplify Eq. 5; yields Eq. 6.

$$0 = b \tag{6}$$

Substitute RHS of Eq. 6 into Eq. 2; yields Eq. 7.

$$\psi(x) = a\sin(kx) \tag{7}$$

LHS of Eq. 4 is equal to LHS of Eq. 7; yields Eq. 8.

$$0 = a\sin(kW) \tag{8}$$

Eq. 9 is an identity.

$$0 = a\sin(n\pi) \tag{9}$$

Eq. 8 is valid when Eq. 9 occurs; yields Eq. 10.

$$kW = n\pi \tag{10}$$

Divide both sides of Eq. 10 by W; yields Eq. 11.

$$k = \frac{n\pi}{W} \tag{11}$$

Substitute RHS of Eq. 11 into Eq. 7; yields Eq. 12.

$$\psi(x) = a\sin(\frac{n\pi}{W}x) \tag{12}$$

Normalization condition is Eq. 13.

$$\int |\psi(x)|^2 dx = 1 \tag{13}$$

Conjugate ψ in Eq. 12; yields Eq. 14.

$$\psi(x)^* = a\sin(\frac{n\pi}{W}x) \tag{14}$$

Swap LHS of Eq. 13 with RHS; yields Eq. 15.

$$1 = \int |\psi(x)|^2 dx \tag{15}$$

Expand $\psi(x)$ in Eq. 15 with conjugate; yields Eq. 16.

$$1 = \int \psi(x)\psi(x)^* dx \tag{16}$$

Substitute LHS of Eq. 12 into Eq. 16; yields Eq. 17.

$$1 = \int_0^W a\sin\left(\frac{n\pi}{W}x\right)\psi(x)^* dx \tag{17}$$

Substitute LHS of Eq. 14 into Eq. 17; yields Eq. 18.

$$1 = \int_0^W a^2\left(\sin\left(\frac{n\pi}{W}x\right)\right)^2 dx \tag{18}$$

Eq. 19 is an identity.

$$(\sin(x))^2 = \frac{1-\cos(2x)}{2} \tag{19}$$

Change variable $\frac{n\pi}{W}x$ to x in Eq. 19; yields Eq. 20.

$$\left(\sin\left(\frac{n\pi}{W}x\right)\right)^2 = \frac{1-\cos\left(2\frac{n\pi}{W}x\right)}{2} \tag{20}$$

Substitute RHS of Eq. 20 into Eq. 18; yields Eq. 21.

$$1 = \int_0^W a^2\frac{1-\cos\left(2\frac{n\pi}{W}x\right)}{2}dx \tag{21}$$

Divide both sides of Eq. 21 by a^2; yields Eq. 22.

$$\frac{1}{a^2} = \int_0^W \frac{1 - \cos\left(2\frac{n\pi}{W}x\right)}{2} dx \tag{22}$$

Expand integrand of Eq. 22; yields Eq. 23.

$$\frac{1}{a^2} = \int_0^W \frac{1}{2} dx - \frac{1}{2}\int_0^W \cos\left(2\frac{n\pi}{W}x\right) dx \tag{23}$$

Eq. 24 is an identity.

$$\int \cos(ax) dx = \frac{1}{a}\sin(ax) \tag{24}$$

Change variable $\frac{2n\pi}{W}$ to a in Eq. 24; yields Eq. 25.

$$\int \cos\left(\frac{2n\pi}{W}x\right) dx = \frac{W}{2n\pi}\sin\left(\frac{2n\pi}{W}x\right) \tag{25}$$

Eq. 26 is an identity.

$$\int a dx = ax \tag{26}$$

Change variable $1/2$ to a in Eq. 26; yields Eq. 27.

$$\int \frac{1}{2} dx = \frac{1}{2}x \tag{27}$$

Substitute LHS of Eq. 27 into Eq. 23; yields Eq. 28.

$$\frac{1}{a^2} = \frac{1}{2}W - \frac{1}{2}\int_0^W \cos\left(2\frac{n\pi}{W}x\right) dx \tag{28}$$

Substitute RHS of Eq. 25 into Eq. 28; yields Eq. 29.

$$\frac{1}{a^2} = \frac{1}{2}W - \frac{1}{2}\frac{W}{2n\pi}\sin\left(\frac{2n\pi}{W}x\right)\Big|_0^W \tag{29}$$

Simplify Eq. 29; yields Eq. 30.

$$\frac{1}{a^2} = \frac{W}{2} \tag{30}$$

Multiply both sides of Eq. 30 by $a^2\frac{2}{W}$; yields Eq. 31.

$$\frac{2}{W} = a^2 \tag{31}$$

Take the square root of both sides of Eq. 31; yields Eq. 32 and Eq. 33.

$$\sqrt{\frac{2}{W}} = a \tag{32}$$

$$-\sqrt{\frac{2}{W}} = a \tag{33}$$

Substitute LHS of Eq. 32 into Eq. 12; yields Eq. 34.

$$\psi(x) = -\sqrt{\frac{2}{W}} \sin\left(\frac{n\pi}{W}x\right) \tag{34}$$

Substitute LHS of Eq. 33 into Eq. 12; yields Eq. 35.

$$\psi(x) = \sqrt{\frac{2}{W}} \sin\left(\frac{n\pi}{W}x\right) \tag{35}$$

Eq. 35 is one of the final equations. Substitute RHS of Eq. 1 into Eq. 2; yields Eq. 36.

$$\frac{d^2}{dx^2}\left(a\sin(kx) + b\cos(kx)\right) = -k^2\left(a\sin(kx) + b\cos(kx)\right) \tag{36}$$

Simplify Eq. 36; yields Eq. 37.

$$a\frac{d^2}{dx^2}\sin(kx) + b\frac{d^2}{dx^2}\cos(kx) = -ak^2\sin(kx) + -bk^2\cos(kx) \tag{37}$$

Simplify Eq. 37; yields Eq. 38.

$$-ak^2\sin(kx) + -bk^2\cos(kx) = -ak^2\sin(kx) + -bk^2\cos(kx) \tag{38}$$

Thus we see that LHS of Eq. 38 is equal to RHS.

References